Introduction to Symplectic Topology: corrigenda

Several readers have pointed out to us various small errors and typos in this book. All are minor except for an error in the statement of Theorem 3.17 on p 94 spotted by David Theret. We thank him as well as everyone else who told us of these errors.

The first part of this note is a list of short corrections. The second part contains some longer revised passages.

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A list of Short Corrections

p 10 line 10: "...such as a pendulum or top..."

p 17 line 7: $J_0 X_{H_t} = \nabla H_t$

p 22 line 20: "...their solutions." line 28: "...function of the variables..."

p. 24 formula (1.19): for consistency with later formulas there should be a $-$ sign in this equation

p. 39, line 5 of Proof: for all $v \in W$

p. 43 On line 14 the text should read: "To prove this, we choose a positive definite and symmetric matrix $P \in \mathrm{Sp}(2n)$ such that...." and on line 21/22: "...choosing P is equivalent to choosing a G-invariant inner product on $\mathbf{R}^{2n}$ that is compatible with ω_0 in the sense that it has the form $\omega_0(\cdot, J\cdot)$ for some ω_0-compatible almost complex structure J."

p 44 line 1/2: "...sends a matrix $U \in \mathrm{SU}(n)$ to..."

p. 47 line 18: $\mu(\Psi) = \frac{1}{2}\sum_t \operatorname{sign}\Gamma(\Psi, t)$

p. 50 line -3: delete repetition of "intersection"

p. 51 line 10: $-\langle \dot{X}(t)u, Y(t)u\rangle$

p. 53 lines 5 and 6 of Proof: replace $\omega_0(\Psi^{\mathrm{T}}v, \Psi^{\mathrm{T}}w)$ by $\omega_0(\Psi^{\mathrm{T}}u, \Psi^{\mathrm{T}}v)$ twice

p. 55 line 10 of Proof: $A\bar{z}_j = -i\alpha_j z_j$

p. 60 line 5,6: replace E^ω by the orthogonal complement $E^\perp$

p. 65 line 10: delete repetition of "that"

p. 71 Exercise 2.66: It would be more clear to say: "there are precisely two" instead of "one nontrivial"

p. 77 last line of Exercise 2.72: $c_1(\nu_{\mathbf{C}P^1}) = 1.$

p. 79, line -10: "Nondegeneracy..."

p. 82 line -1 of proof: $\iota({\psi_t}^* X)\omega = {\psi_t}^*(\iota(X)\omega)$

p. 86, line 2: "By Exercise 2.15, every such..."

p. 89, line 2: "...any vector $v^* \in T_q^* L$ can..."

p. 94 Statement of Theorem 3.17: Delete the last sentence. As David Theret pointed out, it is easy to find a counterexample to this statement unless one requires that the class $[\omega_t] - [\omega_0] \in H^2(M, Q; \mathbf{R})$ is constant.

p. 96 line -4 of proof: replace N by $\mathcal{N}$. last 3 lines of Proof: Name the diffeomorphisms χ_t instead of ϕ_t. Then $\chi_t^* \omega_t = \omega_0 = \omega$ and the desired extension is $\rho_t \circ \chi_t$.

p. 100 line 2: $\phi : \mathcal{N}(L_0) \to V$

p. 103 line -16: "...that $d\alpha$ restricts to..." line -2: Replace Corollary 2.4 by Corollary 2.5

p. 111 line -3: "...symplectization of Q."

p. 113 line -2 of Proof: $\psi^* \omega = e^\theta (d\alpha - \alpha \wedge d\theta)$

p. 115 line 14: "...$f : \mathbf{C}^{n+1} - \mathbf{C}^n \times \{-1\} \to \mathbf{C}^{n+1} - \mathbf{C}^n \times \{-i\}$..."

p. 117 line -6/-5: "...metric $g(u, v) = \langle u, v \rangle$..."

p. 123 line 8: boldface "**(iv)**".

p. 129 line 2: $\lambda : M \to \mathbf{R}$

p 167 line -5: replace "$p_1 \sim p_2$" by "$p_0 \sim p_1$".

p 169 line -3: $T_p \mathcal{O}(p) \subset (T_p(\mu^{-1}(0)))^\omega$.

p. 172 line -10: delete the repetition of "of"

p. 215 line 3 in Lemma 6.31: bracket missing in "$\langle c_1(\nu_\Sigma), [\Sigma] \rangle$".

p. 219 Exercise 6.38: the curve C_3 should be $\{z_1 = {z_2}^3\}$.

p. 221 Lemma 6:40: "...$L(\delta) - L(0)$ is symplectomorphic to the spherical shell $B(\lambda + \delta') - B(\lambda)$ for $\delta' = \sqrt{\lambda^2 + \delta^2}$."

p. 240 last line in definition of $\tilde{\tau}_t(x)$: Replace "$c - \varepsilon \leq f(x)$" by "$f(x) \leq c - \varepsilon$".

p. 253 lines 8,9: "...that, up to diffeomorphism, there is..."

p 265 The proof of Lemma 8.2 must be revised (see below).

p 273 line -3: Replace "$\mathbf{R}^{2n}$" by "$\mathbf{R}^2$".

p 274 line 1/2: Replace "$\mathbf{R}^n$" by "$\mathbf{R}$" and "$\mathbf{R}^{2n}$" by "$\mathbf{R}^2$".

p 303 line 8: ...$F_t = \int_0^t$ ... (not ...$F_t = \int_0^1$ ...);
line -4: "$\Lambda = \mathrm{graph}(dS)$."

p 304 line 2: "...the above action function..."

p 306 line 12: "...where $A = A^T = \partial_x \partial_x \Phi \in \mathbf{R}^{n \times n}$, ..."
line -8: replace "$Ax' - B\xi'$" by "$Ax' + B\xi'$".

p 337 line 14: replace "$\cup a_k$" by "$\cup a_N$".

p 340 line -11: "...every t. A compact invariant set..."

p. 360 line -3: ψ is a symplectic embedding not a symplectomorphism

p. 361 line 16: replace $\overline{c}_G$ by $\overline{w}_G$

p 362 line 1: $c(E) = \pi r_1^2 = w_L(E)$

p 364 line 5 of Proof: the ψ_t are diffeomorphisms not symplectomorphisms

p. 367 lines -7, -3: $\mathcal{L}(\{\phi_t\})$

p 374 line -12: f should be a smooth embedding rather than a diffeomorphism

p. 376 line 6: the restriction of ψ_H to Z_{2c} is called Ψ_H
line -1: $\pi R^2 = c + e + \epsilon$

p. 377 line 3,4 of Exercise 12.22: The text should read: "... symplectic embedding of the ball $B^{2n+2}(r)$ into $B^2(R) \times M$ where $\pi R^2 = e + c/2 + \epsilon$. Using the..."

p 378 line 3 : the supremum and infimum should be taken over $x \in \mathbf{R}^{2n}$

p. 384 line 2 of Proof: $\mathcal{L}_X \omega_0 = \omega_0$

p 390 line -11: the conditions (I), (II), (III).

p 398 line 1: $V_\tau(x_{j+1} + s\xi_{j+1}, y_j + s\eta_j)$

Longer revised passages

page 72: Remark 2.68 (ii)

Several students have pointed out that the sentence "The axioms imply that this integer depends only on the homology class of f" is hard to substantiate. Change this and the rest of Remark 2.68 (ii) to:

"We will see in Exercise 2.75 this integer depends only on the homology class of f. Thus the first Chern number generalizes to a homomorphism $H_2(M, \mathbf{Z}) \to \mathbf{Z}$. This gives rise a cohomology class $c_1(E) \in H^2(M, \mathbf{Z})/\text{torsion}$. There is in fact a natural choice of a lift of this class to $H^2(M, \mathbf{Z})$, also denoted by $c_1(E)$, which is called the **first Chern class**. We shall not discuss this lift in detail, but only remark that in the case of a line bundle $L \to M$ the class $c_1(L) \in H^2(M, \mathbf{Z})$ is Poincaré dual to the homology class determined by the zero set of a generic section."

Then replace the current exercises 2.75 and 2.76 by the following new version:

Exercise 2.75 (i) Prove that every symplectic vector bundle E over a Riemann surface Σ decomposes as a direct sum of 2-dimensional symplectic vector bundles. **Hint:** Show that any such vector bundle of rank > 2 has a nonvanishing section.
(ii) Suppose that Σ is oriented and that the bundle E above extends over a compact oriented 3-manifold X with boundary $\partial X = \Sigma$. Prove that the restriction $E|\Sigma$ has Chern class zero. **Hint:** Use (i) above and look at a section s as in Theorem 2.71.
(iii) Use (i) and (ii) above to substantiate the claim made in Remark 2.68 above that the Chern class $c_1(f^*E)$ depends only on the homology class of f. Here the

main problem is that when $f_*([\Sigma])$ is null-homologous the 3-chain that bounds it need not be representable by a 3-manifold. However its singularities can be assumed to have codimension 2 and so the proof of (ii) goes through. □

This is a new exercise that should go at the very end of Chapter 2.

Exercise Prove that every symplectic vector bundle $E \to \Sigma$ over a Riemann surface Σ which admits a Lagrangian subbundle can be symplectically trivialized. **Hint:** Use the proof of Theorem 2.67 to show that $c_1(E) = 0$. □

This is a revised version of Lemma 8.2.

Lemma 8.2 *The Poincaré section $\Sigma \cap U$ is a symplectic submanifold of M and the Poincaré section map $\psi : \Sigma \cap U \to \Sigma$ is a symplectomorphism.*

Proof: The hypersurface Σ is of dimension $2n - 2$ and the tangent space at p is

$$T_p\Sigma = \{v \in T_pM \,|\, dG(p)v = dH(p)v = 0\}.$$

The condition $\{G, H\} = \omega(X_G, X_H) \neq 0$ shows that the 2-dimensional subspace spanned by $X_G(p)$ and $X_H(p)$ is a complement of $T_p\Sigma$. Now let $v \in T_p\Sigma$ and suppose that $\omega(v, w) = 0$ for all $w \in T_p\Sigma$. Then $\omega(X_H(p), v) = dH(p)v = 0$ and $\omega(X_G(p), v) = dG(p)v = 0$ and hence $v = 0$. Thus the 2-form ω is nondegenerate on the subspace $T_p\Sigma \subset \mathbf{R}^{2n}$.

To prove that ψ is a symplectomorphism we consider the 2-form

$$\omega_H = \omega + dH \wedge dt$$

on $\mathbf{R} \times M$. This is the **differential form of Cartan**. It has a 1-dimensional kernel consisting of those pairs $(\theta, v) \in \mathbf{R} \times T_pM$ which satisfy

$$v = \theta X_H(p).$$

Now let $D \subset \mathbf{C}$ denote the unit disc in the complex plane and let $u : D \to \Sigma$ be a 2-dimensional surface in Σ. We must prove that

$$\int_D u^*\psi^*\omega = \int_D u^*\omega.$$

To see this consider the manifold with corners

$$\Omega = \{(t, z) \,|\, z \in D,\, 0 \leq t \leq \tau(u(z))\}$$

and define $v : \Omega \to \mathbf{R} \times M$ by

$$v(t, z) = \left(t, \phi^t(u(z))\right).$$

Denote $v_0(z) = v(0,z)$ and $v_1(z) = v(\tau(u(z)), z)$. Then ${v_0}^*\omega_H = u^*\omega$ and ${v_1}^*\omega_H = u^*\psi^*\omega$. Moreover, the tangent plane to the surface $v(\mathbf{R}\times\partial D)$ contains the kernel of ω_H. Hence the 2-form $v^*\omega_H$ vanishes on the surface $\mathbf{R}\times\partial D$. Since ω_H is closed it follows from Stokes' theorem that

$$0 = \int_\Omega v^* d\omega_H = \int_{\partial\Omega} \omega_H = \int_D u^*\psi^*\omega - \int_D u^*\omega.$$

Hence ψ is a symplectomorphism. $\square$

This is a revised version of Lemma 12.37 and Exercise 12.38.

Lemma 12.37 *Let H be any Hamiltonian which equals*

$$H_\infty(z) = (\pi+\varepsilon)|z_1|^2 + \tfrac{1}{2}\pi|z_r|^2.$$

for large $|z|$. Then the functional $\Phi_H^\tau : \mathbf{R}^{2nN} \to \mathbf{R}$ satisfies the Palais–Smale condition.

Proof: The Palais–Smale condition asserts that for every sequence $\mathbf{z}^\nu$ in $\mathbf{R}^{2nN}$

$$\left\|\operatorname{grad}\Phi_H^\tau(\mathbf{z}_\nu)\right\|_\tau \to 0 \qquad \Longrightarrow \qquad \sup_\nu \left\|\mathbf{z}_\nu\right\|_\tau < \infty.$$

Suppose otherwise that $\left\|\mathbf{z}_\nu\right\|_\tau \to \infty$. Then, since $\operatorname{grad}\Phi_H^\tau(\mathbf{z}_\nu)$ converges to zero, we claim that all components z_j^ν of $\mathbf{z}_\nu$ must diverge to infinity. Clearly, this will follow if we prove the inequality

$$\min_j |z_j^\nu| \geq \frac{1}{c}\max_j |z_j^\nu| - 1$$

for some constant $c \geq 1$ which is independent of ν. A proof of this is sketched in Exercise 12.38 below. Hence we may assume that the z_j^ν all lie in a region in which $H(z) = H_\infty(z)$. Now consider the sequence

$$\mathbf{w}_\nu = \frac{\mathbf{z}_\nu}{\left\|\mathbf{z}_\nu\right\|_\tau}.$$

This sequence has a convergent subsequence, and it is easy to check that the limit has norm 1 and is a critical point of $\Phi_{H^\infty}^\tau$. But, because the flow of H_∞ has no nonconstant periodic orbits of period 1, the fixed point 0 is the only critical point of this functional. This contradiction proves the lemma. $\square$

Exercise 12.38 This exercise fills in a missing detail in the proof of the above Lemma. Given a vector $\mathbf{z}$ in $\mathbf{R}^{2nN}$ with components $z_j = (x_j, y_j) \in \mathbf{R}^{2n}$ denote by $\zeta_j = (\xi_j, \eta_j) \in \mathbf{R}^{2n}$ the jth component of $\operatorname{grad}\Phi_H^\tau(\mathbf{z})$. Then x_{j+1} is the unique solution of the equation

$$x_{j+1} = F_j(x_{j+1}, \eta_j)$$

where

$$F_j(x_{j+1}, \eta_j) = x_j + \tau \frac{\partial V_\tau}{\partial y}(x_{j+1}, y_j) + \tau \eta_j.$$

Prove that for sufficiently small τ and any η_j the map $x_{j+1} \mapsto F_j(x_{j+1}, \eta_j)$ is a contraction with Lipschitz constant $\alpha = \tau \sup_x \left|\partial^2 V_\tau / \partial x \partial y(x, y_j)\right| < 1$. Deduce that

$$|x_{j+1} - x_j| \le \frac{\tau}{1-\alpha} \left| \frac{\partial V_\tau}{\partial y}(x_j, y_j) + \eta_j \right|.$$

Use this and the inequality

$$|y_{j+1} - y_j| \le \tau \left| \frac{\partial V_\tau}{\partial x}(x_{j+1}, y_j) + \xi_{j+1} \right|$$

to conclude that, if $\|\mathrm{grad}\, \Phi_H^\tau(\mathbf{z})\|_\tau \le 1$ and τ is sufficiently small then

$$|z_{j+1} - z_j| \le \frac{1}{2}(|z_j| + 1).$$

This implies

$$2^{-j}(|z_0| + 1) \le |z_j| + 1 \le 2^j(|z_0| + 1)$$

for $j = 0, \dots, N$. Hence, if $\mathbf{z}_\nu \in \mathbf{R}^{2nN}$ is a sequence with $\|\mathrm{grad}\, \Phi_H^\tau(\mathbf{z}_\nu)\|_\tau \le 1$ and τ sufficiently small such that $\|\mathbf{z}_\nu\|_\tau \to \infty$, then $\|z_{\nu j}\|_{\mathbf{R}^{2n}} \to \infty$ for all j. $\square$

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